Universality in Some Classical Coulomb Systems of Restricted Dimension

P. J. Forrester,[1] **B. Jancovici,**[2] **and G. Téllez**[2]

Coulomb systems in which the particles interact through the d-dimensional Coulomb potential but are confined in a flat manifold of dimension $d-1$ are considered. The actual Coulomb potential acting is defined by particular boundary conditions involving a characteristic macroscopic distance W in the direction perpendicular to the manifold: either it is periodic of period W in that direction, or it vanishes on one ideal conductor wall parallel to the manifold at a distance W from it, or it vanishes on two parallel walls at a distance W from each other with the manifold equidistant from them. Under the assumptions that classical equilibrium statistical mechanics is applicable and that the system has the macroscopic properties of a conductor, it is shown that the suitably smoothed charge correlation function is universal, and that the free energy and the grand potential have universal dependences on W (universal means independent of the microscopic detail). The cases $d = 2$ are discussed in detail, and the generic results are checked on an exactly solvable model. The case $d = 3$ of a plane parallel to an ideal conductor is also explicitly worked out.

1. INTRODUCTION

The present paper is about Coulomb systems, in which the particles interact through the d-dimensional Coulomb potential (or some variant of it)

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[1] Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia; e-mail: matpjf@maths.mu.oz.au.
[2] Laboratoire de Physique Théorique et Hautes Energies, Université de Paris-Sud, 91405 Orsay Cedex, France (Laboratoire associé au Centre National de la Recherche Scientifique-URA D0063); e-mail: janco, tellez@ stat.th.u-psud.fr.

but live in a space of dimension $d-1$ (this is what we call restricted dimension). Examples are particles in a plane interacting through the usual $1/r$ Coulomb potential, or particles on a line interacting through the two-dimensional $-\ln r$ Coulomb potential. We are interested in the classical equilibrium statistical mechanics of some of these systems ("classical" means that the quantum effects are disregarded), with the purpose of exhibiting universal properties ("universal" means independent of the microscopic detail). This universality is closely related to the universality of the macroscopic electrostatics of conductors. Therefore, from the beginning, we assume that the systems under consideration have a macroscopic conducting behavior; this excludes, for instance, a two-component one-dimensional log-gas at a too low temperature, since this gas in then insulating.[1]

The charge correlations have already been shown to be universal for several classes of conducting systems of restricted dimension. In the simple above-mentioned cases,[2] the charge–charge correlation function $S(r)$ is

$$S(r) = -\frac{k_{\mathrm{B}}T}{4\pi^2 r^3} \tag{1.1}$$

in a plane with $1/r$ interactions, and

$$S(r) = -\frac{k_{\mathrm{B}}T}{\pi^2 r^2} \tag{1.2}$$

in a log-gas on a line; k_{B} is Boltzmann's constant, T the temperature, and r the distance. [These expressions (1.1) and (1.2) are only macroscopically valid: the distance r must be large compared to the microscopic scale, and possible oscillations of the correlation function have to be smoothed away.] Another occurrence of universality is for systems with a d-dimensional Coulomb interaction confined in some appropriate d-dimensional domain; then, not only are there universal surface charge correlations,[2] but also the free energy and the grand potential have universal finite-size corrections.[3-5] Here we shall be concerned with infinite flat systems of restricted dimension $d-1$, with some finite-size effect brought in by a boundary condition on the electric potential in the dth dimension. Universal behavior will be found both for the charge correlations and the free energy or the grand potential. For $d=2$, these generic properties will be checked on a solvable model.

For instance, in Section 2, we consider a conducting log-gas on an infinite straight line (the x axis) with the logarithmic interaction modified by the constraint that it be periodic of period W in the transverse direction

y [it has the form (2.2)]. Then, for macroscopic distances, the smoothed charge–charge correlation function is found to be

$$S(x) = -\frac{k_B T \cosh(\pi x/W)}{W^2 \sinh^2(\pi x/W)} \tag{1.3}$$

while the free energy per unit length f and the grand potential per unit length ω exhibit a finite-W correction as $W \to \infty$:

$$f(W) = f(\infty) + k_B T \frac{\pi}{8W} + o(W^{-1}) \tag{1.4a}$$

$$\omega(W) = \omega(\infty) + k_B T \frac{\pi}{8W} + o(W^{-1}) \tag{1.4b}$$

These generic results (1.3) and (1.4) are checked in the special case of a one-component log-gas which is a (partially) exactly solvable model.

Similar results are obtained for a conducting line at a distance W from an ideal conductor in Section 3, and for a conducting line with ideal conductors on each side of it in Section 4. An example of higher dimension is considered in Section 5: a conducting plane at a distance W from an ideal conductor.

2. LOG-GAS ON A LINE WITH TRANSVERSE PERIODIC BOUNDARY CONDITIONS

2.1. The System

The two-dimensional Coulomb interaction between two point charges q and q' located in the xy plane at $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$ is $qq'G_0(\mathbf{r}, \mathbf{r}')$, with $G_0(\mathbf{r}, \mathbf{r}')$ a solution of the Poisson equation

$$\Delta G_0(\mathbf{r}, \mathbf{r}') = -2\pi\delta(\mathbf{r} - \mathbf{r}') \tag{2.1}$$

In the strip $-W/2 \leqslant y$, $y' \leqslant W/2$, a solution of (2.1) with periodic boundary conditions at $y = -W/2$ and $y = W/2$ is

$$G_0(\mathbf{r}, \mathbf{r}') = -\ln\left|\frac{W}{\pi} \sinh\frac{\pi(z - z')}{W}\right| \tag{2.2}$$

where $z = x + iy$ and $z' = x' + iy'$ are the complex coordinates; the factor W/π in (2.2) ensures that, in the limit $W \to \infty$, one recovers the usual two-dimensional Coulomb interaction $-\ln|z - z'|$. We consider some one-dimensional system of charges, on the x axis, with the interaction (2.2),

where now $z = x$ and $z' = x'$; some short-range interaction might also be present. It may be noted that the interaction

$$v(x - x') = -\ln \left| \frac{W}{\pi} \sinh \frac{\pi}{W} (x - x') \right| \qquad (2.3)$$

on the x axis interpolates between a logarithmic interaction $-\ln |x - x'|$ at short distances and a linear interaction (one-dimensional Coulomb interaction) $-(\pi/W) |x - x'|$ at large distances.

The system is assumed to have the properties which characterize a conductor, and to be globally neutral. The period W is macroscopic.

2.2. Electric Potential Correlations

For investigating the charge correlations and the thermodynamics, we first need information about the correlations of the electric potential. Let $\phi(\mathbf{r})$ be the microscopic electric potential created at $\mathbf{r}$ by the charges of the system and let us consider the correlation function $\langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle$. For $\mathbf{r}$ and $\mathbf{r}'$ at macroscopic distances from the x axis, this correlation function can be obtained by the method described in ref. 1, using linear response theory and the conducting behavior assumption, as follows. Let us put an infinitesimal test charge q at $\mathbf{r}'$. Its interaction with the system is described by a Hamiltonian $q\phi(\mathbf{r}')$, and by linear response theory the average potential at $\mathbf{r}$ created by the charges of the system is changed by

$$\delta\phi(\mathbf{r}) = -\beta q \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle^T \qquad (2.4)$$

where $\beta = 1/k_B T$ and $\langle \cdots \rangle^T$ means a truncated statistical average ($\langle AB \rangle^T = \langle AB \rangle - \langle A \rangle \langle B \rangle$); here, however, $\langle \phi(\mathbf{r}) \rangle = 0$, and the mark "truncated" is superfluous. The total potential change at $\mathbf{r}$ is $qG(\mathbf{r}, \mathbf{r}')$, where $G(\mathbf{r}, \mathbf{r}')$ is given by the macroscopic electrostatics of conductors, i.e., G is the solution of the Poisson equation for a point charge q at $\mathbf{r}'$, with the constraint that it is periodic of period W in y and that it vanishes on the x axis $y = 0$:

$$G(\mathbf{r}, \mathbf{r}') = -\ln \left| \frac{\sinh(\pi/2W)(z - z')}{\sinh(\pi/2W)(z - \bar{z}')} \right| \qquad \text{if} \quad yy' > 0 \qquad (2.5a)$$

$$G(\mathbf{r}, \mathbf{r}') = -\ln \left| \frac{\cosh(\pi/2W)(z - z')}{\cosh(\pi/2W)(z - \bar{z}')} \right| \qquad \text{if} \quad yy' < 0 \qquad (2.5b)$$

(G does not vanish for $yy' < 0$, i.e., the conducting line does *not* screen the regions $y > 0$ and $y < 0$ from each other; this is an effect of the periodic

boundary condition which connects these regions across the boundaries at $y = \pm W/2$). That part of the total potential change which is created by the charges of the system is

$$\delta\phi(\mathbf{r}) = q[\, G(\mathbf{r}, \mathbf{r}') - G_0(\mathbf{r}, \mathbf{r}')\,] \tag{2.6}$$

From (2.4) and (2.6), one obtains for the correlation function

$$\beta\langle \phi(\mathbf{r})\, \phi(\mathbf{r}')\rangle^T = G_0(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \tag{2.7}$$

in terms of (2.2) and (2.5).

2.3. Charge Correlations

Let $\sigma(x)$ be the charge per unit length on the x axis and let the macroscopically smoothed charge–charge correlation function be $S(x - x') = \langle \sigma(x)\, \sigma(x')\rangle^T$. Since $2\pi\sigma(x)$ is equal to the discontinuity of the electric field component $-\partial\phi/\partial y$ across the x axis, and since G_0 does not contribute to that discontinuity, one finds from (2.7)

$$\beta S(x - x') = 2 \lim_{y,\, y' \to 0} \left[-\frac{\partial^2 G(\mathbf{r}, \mathbf{r}')}{\partial y\, \partial y'}\bigg|_{yy' > 0} + \frac{\partial^2 G(\mathbf{r}, \mathbf{r}')}{\partial y\, \partial y'}\bigg|_{yy' < 0} \right] \tag{2.8}$$

Using (2.5) in (2.8) gives (1.3).

An alternative derivation of (1.3) can be given by considering only the x-axis system, with the interaction (2.3) given, without any reference to the "outside world." One starts with the assumption that an external infinitesimal linear charge density $q \exp(ikx)$ is perfectly screened (for a wavenumber k small enough, i.e., macroscopic). Therefore the system responds by creating a charge density $-q \exp(ikx)$. The interaction Hamiltonian of the external charge with the system is $q\tilde{v}(k)\, \tilde{\sigma}(-k)$, where $\tilde{v}(k)$ and $\tilde{\sigma}(k)$ are the Fourier transforms of $v(x)$ and $\sigma(x)$, respectively. By linear response theory, $-q = -\beta q\tilde{v}(k)\langle \tilde{\sigma}(-k)\, \tilde{\sigma}(k)\rangle^T$, i.e.,

$$\beta \tilde{S}(k) = \frac{1}{\tilde{v}(k)} \tag{2.9}$$

where $\tilde{S}(k) = \langle \tilde{\sigma}(-k)\, \tilde{\sigma}(k)\rangle^T$ is the Fourier transform of $S(x)$. Our definition of the Fourier transforms is, for instance,

$$\tilde{v}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} v(x) \tag{2.10}$$

Using (2.3) in (2.10) gives (in the sense of distributions)

$$\tilde{v}(k) = \frac{\pi}{k} \operatorname{ctnh} \frac{Wk}{2} \tag{2.11}$$

and, from (2.9), (1.3) follows. It should be noted that, since $\beta \tilde{S}(k) = [\tilde{v}(k)]^{-1}$ vanishes as $k \to 0$, the expression (1.3) of $S(x)$, which is not valid near $x = 0$, has to be supplemented by the prescription that $\int_{-\infty}^{\infty} S(x)\,dx = 0$. The self-part of $S(x)$ is included when this prescription is used.

2.4. Free Energy or Grand Potential

The thermodynamic potential to be considered is the free energy per unit length f if the canonical ensemble is used, or the grand potential per unit length ω if the grand canonical ensemble is used. As a starting point we consider the derivative $\partial f/\partial W$ (at constant densities) or $\partial \omega/\partial W$ (at constant fugacities).

Let us draw some line parallel to the x axis; this line divides the plane into two regions, up, U, and down, D. The derivative $\partial f/\partial W$ or $\partial \omega/\partial W$ is the force per unit length that region U exerts on region D, i.e., the T_{yy} component of the Maxwell stress tensor:

$$\frac{\partial f}{\partial W} = \frac{\partial \omega}{\partial W} = T_{yy'} = \frac{1}{4\pi} \langle E_y(\mathbf{r})^2 - E_x(\mathbf{r})^2 \rangle \tag{2.12}$$

where $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ is the electric field at $\mathbf{r}$ (T_{yy} should be independent of $\mathbf{r}$). Since $\langle \mathbf{E}(\mathbf{r}) \rangle = 0$, we can replace $\langle \cdots \rangle$ by $\langle \cdots \rangle^T$ in (2.12), and using the derivative of (2.7) with respect to $\mathbf{r}$ and $\mathbf{r}'$ (wih the limit $\mathbf{r}' = \mathbf{r}$ taken at the end of the calculation) one finds

$$\beta T_{yy} = \frac{-\pi}{8W^2} \tag{2.13}$$

By integration of (2.12), one obtains (1.4).

Alternatively, (1.4) can be derived by considering only the x-axis system with the interaction $v(x)$. The partition or grand partition function depends on W through $v(x)$, and differentiating it with respect to W gives a statistical average of $\partial v(x)/\partial W$ to be taken with the charge correlation function $S(x)$ [since $\partial v(x)/\partial W$ vanishes at $x = 0$, the small-x regularization of $S(x)$ plays no role here]:

$$\beta \frac{\partial f}{\partial W} = \beta \frac{\partial \omega}{\partial W} = \frac{1}{2} \int_{-\infty}^{\infty} \beta S(x) \frac{\partial v(x)}{\partial W}\,dx = \frac{1}{2} \int_{-\infty}^{\infty} \beta \tilde{S}(k) \frac{\partial \tilde{v}(k)}{\partial W} \frac{dk}{2\pi} \tag{2.14}$$

Using (1.3) and (2.3), or (2.9) and (2.11), in (2.14), one recovers the result $\beta \, \partial f/\partial W = \beta \, \partial \omega/\partial W = -\pi/8 \, W^2$ and (1.4).

2.5. Solvable Model

The one-dimensional one-component plasma with the interaction (2.3) is a solvable model for special value(s) of the temperature. The model consists of particles of charge e, with a particle density η, and a uniform background of charge density $-e\eta$ which ensures overall neutrality.

The correlations are known at the special temperature such that $\Gamma := \beta e^2 = 2$; they are the same ones as for some quantum zero-temperature system.[6] From Eqs. (3.13a) and (3.21a)[3] of ref. 6,

$$\beta S(x, x') = -2 \left[\frac{1}{W} \frac{\hat{l}(x) \, \hat{l}(-x') - \hat{l}(-x) \, \hat{l}(x')}{\theta'_1(0; e^{-2\pi\eta W}) \cdot 2 \sinh(\pi \, |x - x'|/W)} \right]^2 + 2\eta \, \delta(x - x') \tag{2.15}$$

where

$$\hat{l}(x) = e^{-\pi x/2W} \theta_1 \left(\frac{\pi}{4} + \pi\eta x; e^{-\pi\eta W} \right) \tag{2.16}$$

(θ_1 is a Jacobian theta function). The charge–charge correlation function $S(x, x')$ does not depend only on the distance $|x - x'|$ because the system has a crystalline structure. For q small,

$$\theta_1(u; q) \sim 2q^{1/4} \sin u \tag{2.17}$$

For W large, (2.17) can be used in (2.15) and (2.16), giving

$$\beta S(x, x') \sim -\frac{1}{W^2 \sinh^2[\pi(x - x')/W]}$$

$$\times \left[2 \sin^2 \left(\frac{\pi}{4} + \pi\eta x \right) \sin^2 \left(\frac{\pi}{4} - \pi\eta x' \right) e^{-\pi(x - x')/W} \right.$$

$$+ 2 \sin^2 \left(\frac{\pi}{4} - \pi\eta x \right) \sin^2 \left(\frac{\pi}{4} + \pi\eta x' \right) e^{\pi(x - x')/W}$$

$$\left. - \cos 2\pi\eta x \cos 2\pi\eta x' \right], \qquad x \neq x' \tag{2.18}$$

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[3] A factor 2 is missing in the denominator of Eq. (3.21a) in ref. 6.

The macroscopically smoothed $S(x - x')$ is obtained by averaging out the microscopic oscillations in (2.18), i.e., replacing the $\sin^2$ terms by $1/2$ and the cos terms by 0, which gives

$$\beta S(x - x') \sim - \frac{\cosh[\pi(x - x')/W]}{W^2 \sinh^2[\pi(x - x')/W]} \tag{2.19}$$

in agreement with (1.3).

The free energy can be computed at the special temperatures such that $\Gamma = 1, 2, 4$, starting with the known partition function[7] for a finite system of N particles on a line of length L along the x axis, and an interaction which is periodic of period W in the y direction and also periodic of period L in the x direction; on the line, this interaction is

$$v(x - x') = -\ln \left| \theta_1 \left(\frac{\pi(x - x')}{L}; e^{-\pi W/L} \right) \right| + \text{const} \tag{2.20}$$

[with a suitable choice of the constant, (2.20) goes to (2.3) as $L \to \infty$]. Including in the energy the particle–particle, particle–background, and background–background interactions, one finds for the partition function

$$Z_N(\Gamma) = \left[\frac{\pi \theta_1'(0; q)}{L} \right]^{N\Gamma/2} q^{-N^2 \Gamma/8} \left[\prod_{n=1}^{\infty} (1 - q^{2n}) \right]^{-N^2 \Gamma/2} \frac{1}{N!} C_{N\Gamma} \tag{2.21}$$

where $q := \exp(-\pi W/L)$ and $C_{N\Gamma}$ is the configuration integral

$$C_{N\Gamma} := \prod_{l=1}^{N} \int_0^L dx_l \prod_{1 \leq j < k \leq N} \left| \theta_1 \left(\frac{\pi(x_k - x_j)}{L}; q \right) \right|^{\Gamma} \tag{2.22}$$

$C_{N\Gamma}$ has been computed in ref. 7 for $\Gamma = 1, 2, 4$. In the thermodynamic limit $N \to \infty$, $L/L = \eta$ fixed, one finds for the free energy per unit length $f(\Gamma; W) = -\lim L^{-1} \ln Z_N(\Gamma)$

$$\beta f(1; W) = \frac{1}{2} \eta \ln \frac{\eta}{2} + \frac{\pi}{12 W} - \eta \int_0^{1/2} dt \ln \left[\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi W \eta(n^2 + 2nt)}}{n + t} \right] \tag{2.23a}$$

$$\beta f(2; W) = -\eta \ln(2\pi) + \frac{\pi}{8 W} - \eta \sum_{n=1}^{\infty} \ln(1 - e^{-4\pi W \eta n}) \tag{2.23b}$$

$$\beta f(4; W) = -\eta \ln(8\pi^2 \eta) + \frac{\pi}{12 W}$$

$$- \eta \int_0^1 dt \ln \left[\sum_{n=-\infty}^{\infty} (2n + t) e^{-4\pi W \eta(n^2 + nt)} \right] \tag{2.23c}$$

In the large-W limit, in all cases,

$$\beta f(\Gamma; W) \sim \beta f(\Gamma; \infty) + \frac{\pi}{8W} \tag{2.24}$$

$$\beta f(\Gamma; \infty) = \eta \left[\left(1 - \frac{\Gamma}{2}\right) \ln(2\pi\eta) - \frac{\Gamma}{2} \ln \frac{\Gamma}{2} + \frac{\Gamma}{2} + \ln \left(\frac{\Gamma}{2}\right)! - \ln(2\pi) - 1 \right] \tag{2.25}$$

3. LOG-GAS ON A LINE PARALLEL TO AN IDEAL CONDUCTOR

3.1. The System

The region of interest is the half-plane $y > 0$. The electric potential is constrained to vanish on the x axis, i.e., the x axis is an ideal conductor at zero potential. A solution of (2.1) with that boundary condition is

$$G_0(\mathbf{r}, \mathbf{r}') = -\ln \left[\frac{z - z'}{z - \bar{z}'} \right] \tag{3.1}$$

We consider some one-dimensional system of charges, on the line $y = W$, with the corresponding interaction

$$v(x - x') = -\frac{1}{2} \ln \frac{(x - x')^2}{(x - x')^2 + 4W^2} \tag{3.2}$$

plus perhaps some short-range interaction. It may be noted that this interaction interpolates between $-\ln|x - x'|$ at short distances and $2W^2/(x - x')^2$ at large distances. W is a macroscopic distance, and the system is assumed to have the properties of a conductor.

3.2. Correlations

We follow the same steps as in Section 2. Let us first consider the case when the conducting system on the line $y = W$ is kept at zero macroscopic potential. Now G is the solution of the Poisson equation for a point charge q at $\mathbf{r}'$ with the constraint that it vanishes on the lines $y = 0$ and $y = W$:

$$G(\mathbf{r}, \mathbf{r}') = -\ln \left| \frac{\sinh(\pi/2W)(z - z')}{\sinh(\pi/2W)(z - \bar{z}')} \right| \quad \text{if} \quad 0 < y, \ y' < W \tag{3.3a}$$

$$G(\mathbf{r}, \mathbf{r}') = -\ln \left| \frac{z - z'}{z - \bar{z}' - 2iW} \right| \quad \text{if} \quad y, \ y' > W \tag{3.3b}$$

$$G(\mathbf{r}, \mathbf{r}') = 0, \quad \text{if} \quad 0 < y < W, \ y' > W; \ \text{or} \ y > W, \ 0 < y' < W \tag{3.3c}$$

From the analog of (2.8) one now obtains the universal correlation function

$$\beta S(x) = -\frac{1}{2\pi^2 x^2} - \frac{1}{8W^2 \sinh^2(\pi x/2W)} \tag{3.4}$$

An alternative direct derivation of (3.4) uses (2.9) and the Fourier transform

$$\tilde{v}(k) = \frac{\pi}{|k|}(1 - e^{-2W|k|}) \tag{3.5}$$

Here, $\beta \tilde{S}(k) \to 1/2\pi W$ as $k \to 0$, and the small-x modification of (3.4) must be such that $\beta \int_{-\infty}^{\infty} S(x)\,dx = 1/2\pi W$.

3.3. Free Energy or Grand Potential

Still assuming that the line $y = W$ is kept at zero macroscopic potential, we can use (2.7) (with $\langle \cdots \rangle^T = \langle \cdots \rangle$), (3.1), and (3.3a) for computing the T_{yy} component of the stress tensor at some point $\mathbf{r}$ between the ideal conductor and the conducting line. One finds

$$\beta T_{yy} := \frac{\beta}{4\pi} \langle E_y(\mathbf{r})^2 - E_x(\mathbf{r})^2 \rangle = \frac{\pi}{24W^2} \tag{3.6}$$

The W dependence of the free energy or grand potential is given by

$$\frac{\partial f}{\partial W} = \frac{\partial \omega}{\partial W} = T_{yy} \tag{3.7}$$

provided f and ω are properly defined with a Hamiltonian which includes the self-energy interaction $(1/2)q^2 \ln(2W)$ of each particle of charge q with its image. It should be noted that, in the limit $W \to \infty$, this self-energy and the two-body interaction (3.2) generate a well-defined total Hamiltonian containing only a two-body interaction $-\ln|x - x'|$; therefore f and ω are expected to have well-defined limits as $W \to \infty$. From (3.6) and (3.7), one finds the large-W expansions

$$\beta f(W) = \beta f(\infty) - \frac{\pi}{24W} + o(W^{-1}) \tag{3.8a}$$

$$\beta \omega(W) = \beta \omega(\infty) - \frac{\pi}{24W} + o(W^{-1}) \tag{3.8b}$$

with the universal finite-W correction $-\pi/24W$.

Alternatively, (3.8) can be obtained from (2.14) by using either (3.2) and (3.4) [with the integral on x in (2.14) properly regularized at $x = 0$], or (2.9) and (3.5). Since one must keep the self-contribution from the $(1/2) \ln[(x - x')^2 + 4W^2]$ part of (3.2), it is indeed appropriate to use in (2.4) the full correlation function $S(x)$ or $\tilde{S}(k)$ which includes the self-part; when understood as the Fourier transform of $\beta\tilde{S}(k) = 1/\tilde{v}(k)$ in the sense of distributions, (3.4) does represent the full $\beta S(x)$.

3.4. Nonzero Potential Difference

We now consider the more general case when the system on the line $y = W$ is kept at a nonzero macroscopic potential Φ; correspondingly, there is on that line an average linear charge density

$$\sigma = \frac{\Phi}{2\pi W} \tag{3.9}$$

(3.9) is the equivalent of the familiar charge–potential relation in a plane condenser.

Provided one defines the charge correlation function as $S(x) = \langle \sigma(0) \, \sigma(x) \rangle^T$, where the T (truncated) sign is now relevant, the calculation of $S(x)$ is unchanged and (3.4) is still valid.

The free energy and grand potential, however, get additional terms. Now $\langle E_y(\mathbf{r}) \rangle$ is no longer zero, and (3.6) must be replaced by

$$\beta T_{yy} := \frac{\beta}{4\pi} \langle E_y(\mathbf{r})^2 - E_x(\mathbf{r})^2 \rangle$$

$$= \frac{\beta}{4\pi} [\langle E_y(\mathbf{r})^2 - E_x(\mathbf{r})^2 \rangle^T + \langle E_y(\mathbf{r}) \rangle^2]$$

$$= \frac{\pi}{24W^2} + \frac{\beta\Phi^2}{4\pi W^2} = \frac{\pi}{24W^2} + \pi\beta\sigma^2 \tag{3.10}$$

From $(\partial f/\partial W)_\sigma = T_{yy}$ one obtains

$$\beta f(W) \sim \beta f(\infty) - \frac{\pi}{24W} + \beta\pi\sigma^2 W = \beta f(\infty) - \frac{\pi}{24W} + \frac{\beta\Phi^2}{4\pi W} \tag{3.11a}$$

while from $(\partial\omega/\partial W)_\Phi = T_{yy}$ one obtains

$$\beta\omega(W) \sim \beta\omega(\infty) - \frac{\pi}{24W} - \frac{\beta\Phi^2}{4\pi W} \tag{3.11b}$$

It is well known that the macroscopic electrostatic energy $\Phi^2/4\pi W = \pi\sigma^2$ must come with different signs in f and in ω.

3.5. Solvable Model

The one-dimensional one-component plasma on a line parallel to an ideal conductor is a solvable model,[8,9] at the special temperature $\Gamma := \beta e^2 = 2$; it is a special case of the more general models of two-dimensional plasmas with an ideal conductor wall[10] or between two ideal conductor walls.[5] One uses the grand canonical ensemble, with a fixed linear charge density $-e\eta$ for the background and a fugacity ζ which governs the particle density. The distance W between the system and the ideal conductor can have any value (not necessarily large) to start with.

By a simple adaptation of the formalism in previous work,[5,8,9] we obtain for the grand potential per unit length ω (including the background self-energy)

$$\beta\omega = -\operatorname{Tr}\ln(1+K) + 2\pi\eta^2 W \tag{3.12}$$

where K is the continuous matrix

$$K(x, x') = i\zeta \frac{e^{4\pi\eta W}}{x - x' + 2iW} \tag{3.13}$$

Using the Fourier transform which diagonalizes K

$$\tilde{K}(k) = \int_{-\infty}^{\infty} dx'\, e^{ik(x'-x)} K(x, x') = \begin{cases} 2\pi\zeta e^{4\pi\eta W - 2kW} & \text{if } k > 0 \\ 0 & \text{if } k < 0 \end{cases} \tag{3.14}$$

and writing the trace (per unit length) as $(2\pi)^{-1} \int dk$, one finds

$$\beta\omega = -\int_0^\infty \frac{dk}{2\pi} \ln\left[1 + 2\pi\zeta e^{4\pi\eta W - 2kW}\right] + 2\pi\eta^2 W \tag{3.15}$$

For obtaining a large-W expansion ($\eta W \gg 1$) of (3.15), we make the change of variable $4\pi\eta W - 2kW = -u$, split the integral on u into two integrals in the u-ranges $(-4\pi\eta W, 0)$ and $(0, \infty)$, and write

$$2\pi\eta^2 W = -\frac{1}{4\pi W} \int_{-4\pi\eta W}^0 du \ln e^u \tag{3.16}$$

This leads to the still exact expression

$$\beta\omega = -\eta \ln(2\pi\zeta) - \frac{1}{4\pi W} \left[\int_{-4\pi\eta W}^0 du \ln\left(1 + \frac{1}{2\pi\zeta} e^u\right) \right.$$
$$\left. + \int_0^\infty du \ln(1 + 2\pi\zeta e^{-u}) \right] \tag{3.17}$$

Finally, when $\eta W \gg 1$, we can replace the lowest bound of the first integral in (3.17) by $-\infty$ and change u into $-u$, which gives, up to exponentially small terms,

$$\beta \omega \sim -\eta \ln(2\pi\zeta) - \frac{1}{4\pi W} \int_0^\infty du \ln\left[\left(1 + \frac{1}{2\pi\zeta} e^{-u}\right)(1 + 2\pi\zeta e^{-u})\right]$$

$$= -\eta \ln(2\pi\zeta) - \frac{\pi}{24W} - \frac{1}{8\pi W}[\ln(2\pi\zeta)]^2 \tag{3.18}$$

The corresponding particle density is

$$n = -\zeta \frac{\partial}{\partial\zeta}(\beta\omega) \sim \eta + \frac{1}{4\pi W}\ln(2\pi\zeta) \tag{3.19}$$

and the charge density of the system is $e(n - \eta) = (e/4\pi W)\ln(2\pi\zeta)$. The system is neutral when $\zeta = 1/2\pi$. Otherwise, its average electric potential is $\Phi = 2\pi e(n - \eta) W = (e/2)\ln(2\pi\zeta)$ and therefore, since $\beta e^2 = 2$,

$$\zeta = \frac{1}{2\pi} e^{\beta e \Phi} \tag{3.20}$$

(Φ contributes a term $e\Phi$ to the chemical potential, as expected). Using (3.20) in (3.18), with $\beta e^2 = 2$, shows that (3.11b) is verified in the present solvable model. (3.9) is also verified. Furthermore, for $W \to \infty$, $\beta\omega(\infty) = -\eta \ln(2\pi\zeta)$, $n = \eta$ (independent of ζ), and the thermodynamic relation $\beta f = \beta\omega + (\ln \zeta)n$ becomes $\beta f(\infty) = -\eta \ln(2\pi)$, in agreement with (2.25).

The formalism of previous work[5,8,9] also gives the correlation functions in terms of the continuous matrix $g(x, x')$ defined in matrix notation as

$$g = \frac{K}{1 + K} \tag{3.21}$$

The particle density is

$$n = g(0, 0) \tag{3.22}$$

and the charge correlation function is

$$S(x) = -e^2 |g(x, 0)|^2 + e^2 n \delta(x) \tag{3.23}$$

From (3.21) and (3.14), one finds

$$g(x, 0) = \int_0^\infty \frac{dk}{2\pi} e^{ikx} \frac{\tilde{K}(k)}{1 + \tilde{K}(k)} = \int_0^\infty \frac{dk}{2\pi} \frac{e^{ikx}}{1 + (1/2\pi\zeta) e^{2W(k - 2\pi\eta)}} \tag{3.24}$$

In particular,

$$n = g(0, 0) = \frac{1}{4\pi W} \ln(1 + 2\pi \zeta e^{4\pi \eta W}) \tag{3.25}$$

In the case of interest $\eta W \gg 1$, (3.25) takes the form (3.19), and (3.24) can be rewritten as

$$g(x, 0) \sim \int_0^\infty \frac{dk}{2\pi} \frac{e^{ikx}}{1 + e^{2W(k - 2\pi n)}}$$

$$= -\frac{1}{2\pi i x} \frac{1}{1 + e^{-4\pi n W}}$$

$$+ \int_0^\infty \frac{dk}{2\pi} \frac{e^{ikx}}{ix} \frac{W}{2 \cosh^2 W(k - 2\pi n)} \tag{3.26}$$

where an integration by parts has been performed. For $nW \gg 1$, up to exponentially small terms, by extending the last integral in (3.26) to the range $(-\infty, \infty)$, one finds

$$g(x, 0) \sim \frac{1}{2\pi i} \left[-\frac{1}{x} + \frac{e^{2\pi i n x}}{(2W/\pi) \sinh(\pi x/2W)} \right] \tag{3.27}$$

The microscopic correlation function $S(x)$ is obtained by using (3.19) and (3.27) in (3.23). It has oscillations of period n^{-1}, the average interparticle distance. When these oscillations are averaged out, (3.23) with $\beta e^2 = 2$ agrees with the universal form (3.4).

4. LOG-GAS ON A LINE BETWEEN TWO IDEAL CONDUCTORS

4.1. The System

The region of interest is the plane strip $0 \leqslant y \leqslant W$. The electric potential is constrained to vanish on the lines $y = 0$ and $y = W$, i.e., these lines are ideal conductors at zero potential. A solution of (2.1) with that boundary condition is

$$G_0(\mathbf{r}, \mathbf{r}') = -\ln \left| \frac{\sinh(\pi/2W)(z - z')}{\sinh(\pi/2W)(z - \bar{z}')} \right| \tag{4.1}$$

We consider some one-dimensional system of charges, on the line $y = W/2$, with the corresponding interaction

$$v(x - x') = -\ln \left| \tanh \frac{\pi}{2W} (x - x') \right| \tag{4.2}$$

plus perhaps some short-range interaction. Now v interpolates between $-\ln|x - x'|$ and $2\exp[-(\pi/W)|x - x'|]$. Again W is macroscopic and the system is assumed to be a conductor; it is kept at some potential Φ.

4.2. Correlations

Now, using

$$G(\mathbf{r}, \mathbf{r}') = -\ln\left|\frac{\sinh(\pi/4W)(z - z')}{\sinh(\pi/4W)(z - \bar{z}')}\right| \quad \text{if} \quad 0 < y, \quad y' < \frac{W}{2} \quad (4.3a)$$

$$G(\mathbf{r}, \mathbf{r}') = 0 \quad \text{if} \quad 0 < y < \frac{W}{2}, \quad \frac{W}{2} < y' < W; \quad \text{or}$$

$$\frac{W}{2} < y < W, \quad 0 < y' < \frac{W}{2} \quad (4.3b)$$

and using the analog of (2.8), one finds the universal correlation function

$$\beta S(x) = -\frac{1}{W^2 \sinh^2(\pi x/W)} \quad (4.4)$$

Alternatively, one can use (2.9) and the Fourier transform

$$\tilde{v}(k) = \frac{\pi}{k} \tanh\frac{Wk}{2} \quad (4.5)$$

4.3. Free Energy or Grand Potential

Now the stress tensor component T_{yy} is found to be such that

$$\beta T_{yy} = \frac{\pi}{8W^2} + \frac{\beta\Phi^2}{\pi W^2} \quad (4.6)$$

The charge density is

$$\sigma = \frac{2\Phi}{\pi W} \quad (4.7)$$

Thus,

$$\beta f \sim \beta f(\infty) - \frac{\pi}{8W} + \frac{\beta\pi\sigma^2 W}{4} = \beta f(\infty) - \frac{\pi}{8W} + \frac{\beta\Phi^2}{\pi W} \quad (4.8a)$$

and

$$\beta\omega \sim \beta\omega(\infty) - \frac{\pi}{8W} - \frac{\beta\Phi^2}{\pi W} \tag{4.8b}$$

4.4. Solvable Model

The one-component plasma is again a solvable model, with the present boundary conditions, when $\Gamma := \beta e^2 = 2$. The formalism of ref. 5 is applicable, with now

$$K(x, x') = \frac{\pi\zeta}{2W} \frac{e^{\pi\eta W}}{\cosh[(\pi/2W)(x - x')]} \tag{4.9}$$

and

$$\tilde{K}(k) = \pi\zeta \frac{e^{\pi\eta W}}{\cosh Wk} \tag{4.10}$$

Thus

$$\beta\omega = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln\left[1 + \pi\zeta \frac{e^{\pi\eta W}}{\cosh Wk}\right] + \frac{1}{2}\pi\eta^2 W \tag{4.11}$$

For obtaining a large-W expansion of (4.11), we rewrite it as

$$\beta\omega = -\int_0^{\infty} \frac{dk}{\pi} \ln \frac{1 + e^{-2Wk} + 2\pi\zeta e^{W(\pi\eta - k)}}{1 + e^{-2Wk}} + \int_0^{\pi\eta} \frac{dk}{\pi} \ln e^{W(\pi\eta - k)} \tag{4.12}$$

extract from the first integral in (4.12) the factor

$$\int_0^{\infty} \frac{dk}{\pi} \ln(1 + e^{-2Wk}) = \frac{\pi}{24W} \tag{4.13}$$

and regroup the other terms into two integrals on the k-ranges $(0, \pi\eta)$ and $(\pi\eta, \infty)$. After simple changes of variable, neglecting exponentially small terms (which allows us to extend one of the integration ranges), one obtains

$$\beta\omega \sim -\eta \ln(2\pi\zeta) + \frac{\pi}{24W} - \frac{1}{\pi W} \int_0^{\infty} du \ln\left[(1 + 2\pi\zeta e^{-u})\left(1 + \frac{1}{2\pi\zeta} e^{-u}\right)\right]$$

$$= -\eta \ln(2\pi\zeta) - \frac{\pi}{8W} - \frac{1}{2\pi W} [\ln(2\pi\zeta)]^2 \tag{4.14}$$

The corresponding particle density is

$$n = -\zeta \frac{d}{d\zeta}(\beta\omega) \sim \eta + \frac{1}{\pi W}\ln(2\pi\zeta) \tag{4.15}$$

Neutrality still occurs when $2\pi\zeta = 1$, (3.20) is still valid, and (4.8b) is verified by (4.14).

The correlation function is still given by (3.21) and (3.23) now with (4.10). Thus

$$\begin{aligned}
g(x, 0) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{1 + (1/\pi\zeta)\,e^{-\pi\eta W}\cosh Wk} \\
&\sim \int_{0}^{\infty} \frac{dk}{\pi} \frac{\cos kx}{1 + 2e^{-\pi\eta W}\cosh Wk} \\
&\sim \frac{\sin(\pi n x)}{W\sinh(\pi x/W)}
\end{aligned} \tag{4.16}$$

and

$$\beta S(x) = -\frac{1 - \cos(2\pi n x)}{W^2\sinh^2(\pi x/W)} \tag{4.17}$$

When the oscillations of (4.17) are averaged out, the universal expression (4.4) is verified.

5. COULOMB SYSTEM IN A PLANE PARALLEL TO AN IDEAL CONDUCTOR

5.1. The System

The universal properties discussed in Sections 2–4 can be generalized to systems of higher dimension. As an example, in three-dimensional space xyz, we consider a conducting classical system of charges confined in the plane $z = W$ and kept at an average potential Φ, while the plane $z = 0$ is an ideal conductor at potential zero. Such a model might be relevant for describing electrons trapped at the surface of liquid helium in front of an electrode located under the surface.[11] We transpose the derivations and results of Section 3 to the present system.

We denote a position as $\mathbf{r} = (x, y, z) = (\boldsymbol{\rho}, z)$, where $\boldsymbol{\rho} = (x, y)$. In the half-space $z > 0$, the Green function G_0, solution of

$$\Delta G_0(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{5.1}$$

constrained to vanish on the plane $z = 0$ and at infinity, is

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'*|} \qquad (5.2)$$

where $\mathbf{r}'* = (x', y' - z')$ is the image of $\mathbf{r}'$. In the plane $z = W$, the interaction is

$$v(\mathbf{\rho}, \mathbf{\rho}') = \frac{1}{|\mathbf{\rho} - \mathbf{\rho}'|} - \frac{1}{[(\mathbf{\rho} - \mathbf{\rho}')^2 + 4W^2]^{1/2}} \qquad (5.3)$$

5.2. Correlations

The Green function which vanishes on both the $z = 0$ and $z = W$ planes is

$$G(\mathbf{r}, \mathbf{r}') = \sum_{n = -\infty}^{\infty} G_0(\mathbf{r}, \mathbf{r}' + 2nW\mathbf{u}) \qquad \text{if} \quad 0 < z, \ z' < W \qquad (5.4a)$$

where $\mathbf{u}$ is the unit vector along the z axis,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{[(\mathbf{\rho} - \mathbf{\rho}')^2 + (z + z' - 2W)^2]^{1/2}} \qquad \text{if} \quad z, z' > W \qquad (5.4b)$$

$$G(\mathbf{r}, \mathbf{r}') = 0 \qquad \text{if} \quad 0 < z < W, \ z' > W; \quad \text{or} \quad z > W, \ 0 < z' < W \qquad (5.4c)$$

The potential correlation function is given by (2.7), and from the analog of (2.8) one finds for the surface charge $\sigma(\mathbf{r})$ correlation function in the plane $z = W$

$$\beta S(\mathbf{\rho} - \mathbf{\rho}') := \beta \langle \sigma(\mathbf{\rho}) \sigma(\mathbf{\rho}') \rangle^T = -\frac{1}{4\pi^2} \sum_{n=0}^{\infty} \frac{(\mathbf{\rho} - \mathbf{\rho}')^2 - 2(2nW)^2}{[(\mathbf{\rho} - \mathbf{\rho}')^2 + (2nW)^2]^{5/2}} \qquad (5.5)$$

At short distances, $\beta S \sim -1/4\pi^2(\mathbf{\rho} - \mathbf{\rho}')^3$, in agreement with the formula (1.1) for a plane alone; at long distance, the Euler–MacLaurin summation formula gives $\beta S \sim -1/8\pi^2(\mathbf{\rho} - \mathbf{\rho}')^3$.

An alternative direct derivation of (5.5) uses (2.9) and the two-dimensional Fourier transform

$$\tilde{v}(k) = \frac{2\pi}{|\mathbf{k}|} (1 - e^{-2W|\mathbf{k}|}) \qquad (5.6)$$

5.3. Free Energy or Grand Potential

In the region $0 < z < W$, using (2.7) gives for the T_{zz} component of the Maxwell stress tensor the force per unit area,

$$T_{zz} := \frac{1}{4\pi} \left\langle E_z^2 - \frac{1}{2} \mathbf{E}^2 \right\rangle = \frac{k_B T \zeta(3)}{8\pi W^3} + \frac{\Phi^2}{8\pi W^2} \qquad (5.7)$$

where $\zeta(3) = 1.202...$ is a value of the Riemann zeta funtion. The last term of (5.7) is the standard attractive force between the plates of a plane condenser. The (usually much smaller) thermal term $k_B T \zeta(3)/8\pi W^3$ can be obtained as the classical limit of the celebrated Lifshitz theory[12] of van der Waals-type forces between macroscopic bodies.[4]

By integrating $(\partial f/\partial W)_\sigma = (\partial \omega/\partial W)_\Phi = T_{zz}$, one obtains

$$\beta f(W) = \beta f(\infty) - \frac{\zeta(3)}{16\pi W^2} + \frac{\Phi^2}{8\pi W} \qquad (5.8a)$$

$$\beta \omega(W) = \beta f(\infty) - \frac{\zeta(3)}{16\pi W^2} - \frac{\Phi^2}{8\pi W} \qquad (5.8b)$$

Alternatively the thermal term of (5.8) can be obtained by using (2.9), the analog of (2.14), and (5.6).

For the d-dimensional analog of the present system, the thermal part of the force per unit area is found to be

$$\frac{k_B T(d-1) \, \Gamma(d/2) \, \zeta(d)}{\pi^{d/2}(2W)^d} \qquad (5.9)$$

6. CONCLUSION

The occurrence of universal properties in conducting classical Coulomb systems is especially visible in systems of restricted dimensionality. The simplest, already known, example is the smoothed charge–charge correlation function $S(r)$ for particles in a plane with $1/r$ interactions, as given by Eq. (1.1). If (1.1) is understood as the two-dimensional Fourier transform of $k_B T \, |\mathbf{k}|/2\pi$ in the sense of distributions, the prescription for regularizing its integral is

$$\int S(r) \, d^2\mathbf{r} = -\frac{k_B T}{4\pi^2} \int \frac{d^2\mathbf{r}}{r^3} = 0 \qquad (6.1)$$

and therefore (1.1) represents the smoothed full $S(r)$, including its self-part term, since (6.1) correctly expresses the screening rule.

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[4] Our result agrees with Eq. (5.5) of ref. 12, with $\varepsilon_0 = \infty$ and the integral evaluated exactly.

In the present paper, we have derived universal smoothed charge correlation functions for other geometries involving some boundary conditions for the electric potential outside the system of charges. These boundary conditions involve some macroscopic length scale W, and we have exhibited universal W dependences of the free energy and the grand potential.

In Sections 2–4, we have considered one-dimensional systems with a two-dimensional Coulomb interaction, because, for these systems, we had at hand exactly solvable models on which we could test our generic results. However, similar generic results can be easily derived for any dimension, and in Section 5 we gave an example involving the usual three-dimensional Coulomb law.

A by-product of Section 4 is a new one-dimensional solvable model: particles on a line, interacting through the potential $-e^2 \ln |\tanh[(\pi/2W)(x - x')]|$, without any background or self-energy. When $\Gamma := \beta e^2 = 2$, the correlations and the thermodynamics can be obtained exactly, by a minor adaptation of Section 4.

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